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OPTIMUM EVASION VERSUS SYSTEMATIC SEARCH

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

This study is a contribution to research conducted by The RAND Corporation in the field of game theory. The formulation presented in this Memorandum has theoretical application in both military and civilian situations where searcher and evader tactics are involved.

SUMMARY

The solution of the following hide and seek game is presented: At each move Player I, the evader, is allowed to hide in one room, while Player II, the searcher, is allowed to search some given number of rooms. The restriction is made that Player II searches without repetition, that is, he is never allowed to return to a room he has previously searched. It is shown that if the payoff to Player I is any increasing function of the number of moves before capture, his best strategy is also never to return to a room in which he has previously hidden. A formula for the value of the game is presented.

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CONTENTS

PREFACE iii

SUMMARY v

ACKNOWLEDGMENTS vii

INTRODUCTION 1

FORMAL DEFINITIONS AND A STATEMENT OF THE THEOREMS 2
Theorem 1 (Special Case) 5
Theorem 2 (Special Case) 6
Value Theorem (Special Case) 7
Theorem 3 (General Case) 7

PROOFS 8
Main Lemma 9
Corollary 10
Proof of Theorems 1 and 2 11

REFERENCE 13

INTRODUCTION

The common situation of search versus evasion can be described precisely in the language of game theory: Player I, the evader, and Player II, the searcher, each choose an integer from 1 through N (each integer representing one of N separate rooms). This process is repeated until either a match occurs (i.e., II finds I) or the game has run some preselected number n of moves. In the first case there is no payoff, while in the second case II pays I one unit. If no restrictions are placed on the strategies of the players the solution of this game is trivial--each player chooses randomly with equal probability from the complete set $\{1, 2, \ldots N\}$ at each move, and the value of the game is $(1-1/N)^n$.

This paper investigates the solution of the same game under the restriction that II's search be systematic--after Player II visits a room he may not return to that room until he has searched through the N-1 others. It is natural in this case to insist that n be equal to or less than N. II's possible pure strategies are represented by the permutations of n out of the N integers. (A permutation is defined as an ordered set without repetition.) From the symmetry it is easily seen that II's optimum strategy is to choose the $[N]_n = (N)(N-1)(N-2)\ldots(N-n+1)$ permutations randomly and with equal probability.

The main result of this paper is a proof that under these circumstances I's optimum strategy is exactly the same as II's; I randomly chooses n <u>different</u> integers, never making use of his option to return to a room he has already visited. A simple expression for the value of the game is derived, also yielding the expected payoff for any other

strategy Player I may choose. The theorems are actually proved in much greater generality--when II is allowed to search more than one room per move, when the payoff to I is any increasing function of his survival time, and conditionally, when for any reason another rule has been followed up to a certain point. The universality of the "no return rule" is of course very useful to Player I, who can play optimally with practically no information on the actual structure of the game.

The condition of systematic search is often encountered in practice, either because of ignorance, physical inconvenience, or ulterior motives on the searcher's part. The watchman, the detective, and the air reconnaissance team anxious to cover as much ground as possible are all likely to search non-repetitively. If the evader has no additional information on the searcher's pattern, then it is sensible for him to play the "no return rule". For in this case the searcher's optimum strategy, all permutations equally likely, coincides with the natural "ignorance" distribution.

FORMAL DEFINITIONS AND A STATEMENT OF THE THEOREMS

In the most general situation discussed in this paper Player II will be allowed to search M_1 rooms on his first move, M_2 on the second, ... and M_n on the nth, where the M_i are fixed constants such that

$$M_1 + M_2 \ldots + M_n \equiv N_0 \leq N.$$

(He searches without repetition, as explained in the introduction.) Player I is allowed to hide in only one room per move but he may return to rooms he has already visited if he so desires. If Player II first finds Player I on the j_0th move, I receives a payoff of $P(j_0)$ units,

(Pn+1) being the payoff if no detection occurs, $P(1) \leq P(2) \cdots \leq P(n+1)$.

Formally we denote II's selection of N_0 different rooms by

$$(r_{11}, r_{12} \cdots r_{1M_1}, r_{21} \cdots r_{2M_2}, \cdots r_{n1} \cdots r_{nM_n}) ,$$

-a permutation of size $N_0 = M_1 + M_2 \ldots + M_n$ from the first N positive integers. Player I selects an n-tuple

$$(R_1, R_2 \ldots R_n)$$

from the first N positive integers, repetitions being allowed. R_j is Player I's hiding place and $\{r_{j1}, r_{j2} \ldots r_{jM_j}\}$ the rooms searched on the jth move. If j_0 is the smallest value of j such that

$$R_j \in \{r_{j1}, r_{j2} \cdots r_{jM_j}\}$$

Player II pays Player I $P(j_o)$ units. If there is no such value of j, I receives $P(n+1)$ units, where the $P(j)$ form a non-decreasing sequence. This situation will be referred to as the general case.

When

$$M_1 = M_2 = M_3 \ldots = M_n \equiv M$$

and

$$P(1) = P(2) \ldots = P(n) = 0$$

$$P(n+1) = 1,$$

we shall say we are in the special case. (The special case with M = 1 the game discussed in the second paragraph of the introduction.)

Assume first that Player II plays "random permutations," that is II chooses among the $[N]_{N_0}$ possible permutations randomly and with

equal probability. The main conclusion of this paper is that Player I's best counter-strategy is then also to play random permutations; Player I selects the n-tuple $(R_1, R_2 \ldots R_n)$ randomly and with equal probability from the $[N]_n$ possible permutations of n out of the N integers. But when I plays in this manner, any strategy II may choose yields the same expected payoff, which implies that random permutations is the optimum strategy for both players. <u>In the sequel it is always assumed that Player II, but not necessarily Player I, behaves in this way</u>.

Define $R(n_1, n_2 \ldots n_k)$ as the set of n-tuples $(R_1, R_2 \ldots R_n)$ involving k distinct integers with frequencies $n_1, n_2, \ldots . n_k$ respectively, where necessarily

$$n_1 + n_2 + \ldots . n_k = n$$

Each element of $R(n_1, n_2 \ldots . n_k)$ is a pure strategy for Player I. It is important to note that in the <u>special case</u>, all of these pure strategies (and hence any probabilistic mixture of them) give Player I the same probability of escaping detection for the duration of the game when Player II is playing random permutations. We denote this probability by

$$V(n_1, n_2 \ldots n_k)$$

For convenience we call the set of all mixed strategies for Player I involving only the elements of $R(n_1, n_2 \ldots n_k)$ "the strategy $S(n_1, n_2 \ldots n_k)$" (or, if no confusion is possible, simply "the strategy S".) Thus

$$V(n_1, n_2 \ldots n_k) \equiv V(S)$$

is the expected payoff versus random permutations of strategy

$$S(n_1,n_2\ldots n_k) \equiv S$$

in the special case.

(As a specific example of these definitions let $N=8$, $n=3$, $n_1=1$, $n_2=2$. Then $(R_1,R_2,R_3) = (2,7,7)$ and $(R_1,R_2,R_3) = (3,5,3)$ are both elements of $R(n_1,n_2)$. The pure strategy for player I "room 2 first move, room 7 second move, room 7 third move" is a member of $S(n_1,n_2)$, as is the mixed strategy "20% of the time room 2 on the first move, room 7 on the second, room 7 on the third; 80% of the time room 3 on the first move, room 5 on the second, room 3 on the third.")

For the purpose of evaluating $V(n_1,n_2\ldots.n_k)$ we can use any element of $R(n_1,n_2\ldots n_k)$ to realize $S(n_1,n_2\ldots n_k)$. A particularly useful choice will be

$$(R_1,R_2\ \ldots R_n) \equiv (\underbrace{1,1,\ldots..1}_{n_1},\ \underbrace{2,2,\ldots..2}_{n_2},\ldots\ldots\ldots\ldots..\underbrace{k,k,\ldots..k}_{n_k})$$

Given

$$S \equiv S(n_1,n_2\ldots n_h\ldots n_m\ldots n_k),$$
$$\text{where } n_h < n_m,$$

the two strategies

$$S' \equiv S(n_1,n_2\ldots n_{h-1},n_h+1,n_{h+1}\ldots n_{m-1},n_m-1,n_{m+1}\ldots n_k)$$

and

$$S'' \equiv S(n_1,n_2\ldots n_h\ldots n_{m-1},n_m-1,n_{m+1}\ldots n_k, 1)$$

are both said to be <u>simply derived</u> from S. S_2 is derived from S_1 if it can be attained from S_1 by a sequence of simple derivations.

<u>Theorem 1 (special case)</u>: <u>If S_2 is derived from S_1 then $V(S_2) \geq V(S_1)$.</u>

Among all strategies $S(n_1,n_2 \dots n_k)$ such that $n_1 + n_2 \dots + n_k = n$, the "no return" rule $\bar{S} \equiv \underbrace{S(1,1 \dots 1)}_{n}$ maximizes $V(S)$.

Define

$$S'(n_1', n_2' \dots n_{k'}') \prec S(n_1, n_2 \dots n_k)$$

if $\qquad k' \leq k$

and $\qquad n_i' \leq n_i \qquad i = 1,2 \dots k'$.

A representative realization of S is the permutation

$$(R_1,R_2 \dots R_n) = (\underbrace{1,1 \dots 1}_{n_1'},\underbrace{2,2 \dots 2}_{n_2'} \dots \underbrace{k',k' \dots k'}_{n_{k'}'},\underbrace{1,1 \dots 1}_{n_1 - n_1'} \dots \underbrace{k',k' \dots k'}_{n_{k'} - n_{k'}'},\underbrace{k'+1, \dots k'+1}_{n_{k'}+1} \dots \underbrace{k,k \dots k}_{n_k})$$

which shows that Player I may think of playing strategy S' first and then, if he is not detected by Player II, completing it to strategy S. The conditional probability that Player I will escape detection playing strategy S, given that he has already played S' without detection is defined as $V(S \mid S')$. (A formal definition of the conditional expectation is postponed until the next section.)

<u>Theorem 2 (special case)</u>: <u>Suppose $S_o \prec S_1$, $S_o \prec S_2$, and S_2 is derived from S_1. Then $V(S_2 \mid S_o) \geq V(S_1 \mid S_o)$. Given any strategy $S_o \equiv S(n_1', n_2', \dots n'_{k'})$, where</u>

$$n_1' + n_2' + \dots + n_{k'}' \equiv n' \leq n,$$

<u>Let</u>
$$\bar{S}_o = S(n_1', n_2', \dots n_k', \underbrace{1,1, \dots 1}_{n - n'}).$$

<u>Then $V(\bar{S}_o \mid S_o) = \max V(S \mid S_o)$ among all strategies $S(n_1,n_2, \dots n_k)$ such that $S_o \prec S$ and $n_1 + n_2 + \dots + n_k = n$.</u> (In other words, Player I should begin playing the "no return rule" as soon as he is allowed to do so.)

Given strategy $S(n_1, n_2, \ldots n_k)$, define

$$\sigma_1 = \sum_{i=1}^{k} n_i$$

$$\sigma_2 = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} n_i n_j$$

$$\vdots$$

$$\sigma_k = n_1 n_2 \ldots n_k.$$

(the σ_i are the elementary symetric functions of $n_1, n_2 \ldots n_k$.)

Value Theorem (Special Case)

$$V(n_1, n_2 \ldots n_k) = 1 - \frac{M\sigma_1}{[N]_1} + \frac{M^2\sigma_2}{[N]_2} \cdots + \frac{(-M)^k\sigma_k}{[N]_k} .$$

In particular for

$$\bar{S} = S(\underbrace{1, 1 \ldots 1}_{n}),$$

$$V(\bar{S}) = \sum_{h=0}^{n} \frac{(-M)^h}{N!} \frac{n! \; (N-h)!}{(n-h)!h!}$$

which is the value of the game (attained when both players play random permutations.)

Theorem 3 (general case)

Random permutations is the optimum strategy for both players.

More elaborate results are obtainable in the general case, but they will not be investigated here. It should be noted that theorem 3 holds true regardless of the constants $M_1, M_2 \ldots M_n$, and hence they need not be assumed known to Player I.

PROOFS

The second halves of Theorems 1 and 2 are simple consequences of the first statements. To prove these it is sufficient to consider only the subcase M = 1, since the strategy

$$S \equiv S(n_1, n_2 \ldots n_k)$$

for general M is equivalent to the strategy

$$S^* \equiv S(n_1M, n_2M, \ldots n_kM)$$

for M = 1. Furthermore, using the obvious notation, if S_2 is derived from S_1 then S_2^* is derived from S_1^*, and if $S_0 \prec S$ then $S_0^* \prec S^*$.

In the special subcase with M = 1 we change notation slightly and denote Player II's randomly selected permutation by

$$(r(1), r(2) \ldots r(n)).$$

as commented previously, a strategy $S(n_1,n_2\ldots n_k)$ for Player I can be realized by the n-tuple

$$(R_1,R_2\ldots R_n) = (\underbrace{1,1\ldots 1}_{n_1}, \underbrace{2,2\ldots 2}_{n_2} \ldots \underbrace{k,k\ldots k}_{n_k}).$$

Define

$$A_i \equiv \left\{r(1+\sum_{j=1}^{i-1} n_j),\ r(2+\sum_{j=1}^{i-1} n_j) \ldots r(n_i + \sum_{j=1}^{i-1} n_j)\right\} \quad i = 1,2\ldots k$$

$$A_i \equiv \emptyset \qquad\qquad i > k.$$

A_i is the set of rooms Player II searches while Player I is hiding in room i under the strategy $(R_1,R_2\ldots R_n)$ above. In what follows it is assumed that I plays this fixed pure strategy, while II plays random

permutations. Then the sets A_i are functions of the random permutation $(r(1), r(2), \ldots\ldots r(n))$, and the event of no capture,

$$r(j) \neq R_j \quad j = 1, 2, \ldots\ldots n$$

is equivalent to the event

$$i \notin A_i, \quad i = 1,2 \ldots k.$$

<u>Main Lemma</u>

$$\text{Let } p_h = P(h \in \bigcup_1^k A_i \mid i \notin A_i, \ i = 1,2 \ldots k)$$

$$h = 1,2 \ldots N.$$

Then if $n_1 \geq n_2 \geq \ldots \geq n_k$,

$$p_1 \leq p_2 \leq \cdots \leq p_{k+1} = p_{k+2} = \cdots = p_n.$$

Proof: Set $n_h = 0$ for $h > k$ and suppose that for some m and h, $n_m > n_h$.

Define

$$B \equiv \left\{ r(1+\sum_{j=1}^{m-1} n_j),\ r(2+\sum_{j=1}^{m-1} n_j) \ldots r(n_m - n_h + \sum_{j=1}^{m-1} n_j) \right\}$$

and $\quad \bar{A}_m \equiv A_m - B$

$\bar{A}_i \equiv A_i \quad , \quad i \neq m.$

Then

$$p_h \cdot P(i \notin A_i \quad i = 1,2 \ldots k) =$$

$$= P(h \in B,\ m \notin B,\ i \notin \bar{A}_i \quad i = 1,2 \ldots k)$$

$$+ P(h \notin B,\ m \notin B,\ h \in \bigcup_{i=1}^{k} \bar{A}_i,\ i \notin \bar{A}_i \quad i = 1,2 \ldots k)$$

$$p_m \cdot P(i \notin A_i \quad i = 1,2 \ldots k)$$

$$= P(h \in B,\ m \notin B,\ i \notin \bar{A}_i \quad i = 1,2 \ldots k,\ m \in \bigcup_a^k A_i)$$

$$+ P(h \notin B,\ m \notin B,\ m \in \bigcup_{i=1}^{k} \bar{A}_i,\ i \notin \bar{A}_i \quad i = 1,2 \ldots k).$$

The second terms on the right sides of these two equations are equal by symmetry, and comparing the first terms verifies the lemma.

Given

$$S'(n_1',\ n_2' \ldots n_{k'}') \prec S(n_1,\ n_2 \ldots n_k)$$

define

$$A_i' = \left\{ r(1+\sum_{j=1}^{i-1} n_j),\ r(2+\sum_{j=1}^{i-1} n_j) \ldots r(n_i' + \sum_{j=1}^{i-1} n_j) \right\} \quad i = 1,2 \ldots k'.$$

The value and conditional value are formally defined by

$$V(S) \equiv V(n_1 \ldots n_k) \equiv P(i \notin A_i\ ,\quad i = 1,2 \ldots k)$$

$$V(S \mid S') \equiv P(i \notin A_i,\quad i = 1,2 \ldots k \mid i \notin A_i' \quad i = 1,2 \ldots k').$$

Finally, for $n_h < n_m$ (n_h may equal zero), let

$$S_0 \equiv S(n_1, n_2 \ldots n_h \ldots n_m - 1 \ldots n_k)$$

$$S_1 \equiv S(n_1, n_2 \ldots n_h \ldots n_m \ldots n_k)$$

$$S_2 \equiv S(n_1, n_2 \ldots n_h + 1 \ldots n_m - 1 \ldots n_k).$$

<u>Corollary</u> $\quad V(S_2 \mid S_0) \geq V(S_1 \mid S_0).$

Proof:

$$V(S_1 \mid S_0) = 1 - \frac{(1 - p_m)}{N - n + 1}$$

$$V(S_2 \mid S_0) = 1 - \frac{(1 - p_h)}{N - n + 1}\ .$$

Applying the lemma proves the corollary.

Proof of Theorems 1 and 2

Applying the definitions, if $S' \prec S$ then

$$V(S) = V(S')\, V(S \mid S').$$

Thus the corollary implies that $V(S_2) \geq V(S_1)$, and iterating this result proves Theorem 1. The equality above also shows that Theorem 1 implies Theorem 2.

Define

$$T_i \equiv \left\{ (r(1), r(2) \ldots r(n)) \mid i \in A_i \right\} \quad i = 1,2 \ldots k$$

(the elements of T_i are permutations of size n from the first N positive integers) so that the event

$$i \notin A_i \qquad i = 1,2 \ldots k$$

is equivalent to the event

$$(r(1), r(2) \ldots r(n)) \notin \bigcup_{i=1}^{k} T_i \ .$$

By direct evaluation

$$P\,(T_{i_1}\ T_{i_2} \cdots T_{i_h}) = \frac{n_{i_1}\ n_{i_2} \cdots n_{i_h}}{[N]_h} \ .$$

Applying a well-known formula[(1)] yields the value theorem for $M = 1$, and the remark at the beginning of this section extends the result to general M.

In the general case let $(R_1, R_2 \ldots R_n)$ be any element of $R(n_1, n_2 \ldots n_k)$, m any integer that appears more than once in $(R_1, R_2 \ldots R_n)$, and h the largest integer such that

$$R_h = m.$$

It is a simple consequence of the main lemma that the n-tuple

$$(R_1, R_2 \ldots R_{h-1}, R_h', R_{h+1} \ldots R_n)$$

yields a greater expected payoff than $(R_1, R_2 \ldots R_n)$ versus random permutations whenever

$$R_h' \notin \{R_1, R_2 \ldots R_n\}.$$

Theorem 3 follows by induction.

REFERENCE

1. Parzen, E., Modern Probability Theory and its Applications, Wiley and Sons, New York, 1960, pp. 76-77.